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The axisymmetric flow in a rotating annulus due to a horizontally applied temperature gradient

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The axisymmetric flow of fluid, confined within a rapidly rotating annulus and caused by a horizontally applied temperature gradient, is discussed under the assumptions of small thermal Rossby number and small Taylor number. It is initially assumed that the transfer of heat is purely conductive, and the flow is calculated for when the upper fluid surface is either rigid or free. The analysis of the problem in which the upper surface is rigid is extended so as to allow convective as well as conductive transfer of heat.

1. Introduction

During recent years there has been considerable experimental and theoretical interest in flows driven in a rotating annulus by an applied temperature gradient. Phenomena similar to ones observed in the lower atmosphere arise in laboratory investigations of these flows, and can be studied under controlled conditions. Generally, the annulus has a rectangular cross-section, and is rotated about its axis which is vertical. Its two vertical surfaces are held at constant but differing temperatures, while the horizontal surfaces are insulating. Under these conditions, one of four different flow régimes can occur. The flow may consist of a steady axisymmetric circulation, there may be a wave system which is steady in some rotating frame of reference, the flow may perform a regularly repeating fluctuation, or it may be irregular. The present work is concerned with the first and simplest of these flow régimes. The axisymmetric régime arises for a wide range of magnitudes of the relevant physical parameters, but we shall be concerned with it for the case of rapid rotation and small thermal Rossby number, so that the Coriolis term is dominant in the fluid acceleration, and small Taylor number, so that viscous effects are confined to narrow boundary layers at the surface of the annulus. The precise conditions under which the solutions presented below are valid become apparent only after detailed investigation, but they all lie in the so-called 'lower symmetrical' part of the parameter range for which axisymmetric flow occurs. (For an account of recent experimental work in this field, see Fowlis & Hide (1965), in which a number of other references are given.)

In spite of its relative simplicity, comparatively little theoretical work has been concerned with the axisymmetric régime. There have been more theoretical investigations of the steady wave régime in the form of stability analyses of an axisymmetric flow, though the basic axisymmetric flow used for these analyses has generally been a reasonable one, rather than an exact solution of the flow equations. It is the aim of the present work to help provide an understanding of the axisymmetric flow régime at least for a certain range of physical parameters, and thus to provide a basis for further theoretical work investigating stability.

An analysis of the axisymmetric flow under the above-mentioned conditions of small thermal Rossby number and small Taylor number, together with the assumption that the heat transfer is purely conductive, was given by Robinson (1959). The flow in the interior of the annulus was found to consist of a vertically sheared zonal flow around the annulus. There is also a much weaker meridional circulation in a cross-section of the annulus through its axis. This circulation is carried through thin boundary layers at the walls of the annulus. On the horizontal surfaces these are Ekman boundary layers, which, as is typical for rapidly rotating flow problems, are crucial for determining the properties of the flow throughout the interior of the annulus. The boundary layers on the vertical side walls on the other hand are of only local importance and adjust the remainder of the flow to the side-wall boundary conditions. The analysis of these side boundary layers is more complicated than that of the Ekman layers. Though Robinson derived a solution for the side layers, he cast some doubt on the validity of his solution in a footnote added in proof (p. 611). This footnote suggested that, in the light of an analysis of a related problem by Stewartson (1957), the boundary layers on the side walls are of a more complicated character than that deduced for them in the paper itself, and have a double structure with one boundary layer inside another.

The problem described above is re-examined in the present paper, the flow in the side-wall boundary layers is calculated, and the uncertainty raised by the footnote resolved. It is found that the side boundary layers do not have the double structure suggested by the footnote, but are of the form Robinson originally supposed. This does not vindicate Robinson's original determination of them however, which is in error as no account is taken of the problem of matching the side layers onto Ekman layers at the top and bottom of the annulus. Only when this matching is done can the solution be fully determined.

The same problem, apart from the single change of supposing the top surface of the fluid in the annulus to be free rather than a rigid boundary, is discussed next. This change introduces several interesting modifications, and is of relevance as the upper fluid surface is free in much of the experimental work. Again the flow in the interior of the annulus is a vertically sheared zonal flow, and there is again a weak meridional circulation around the circumscribing boundary layers, though this is weaker by an order of magnitude from what it is when the upper surface is rigid. This is due to the weak character of an Ekman layer at a free surface (Hide 1964). There are larger meridional circulations in each side layer though these never penetrate into the interior. Actually, these also occur in the problem with rigid upper surface, though the disparity in strength is not then so marked. Another feature of the flow when the upper surface is free is that the side boundary layers do have a double structure.

In all the work that has been described so far, the transfer of heat has been supposed purely conductive. The final section is concerned with extensions of the preceding analysis to discuss the modifications introduced when some account is taken of heat convection. The natural way to do this is to calculate by an iterative process the corrections to the solutions derived earlier that are produced when convective effects are small but not entirely negligible. Without calculating these successive corrections in detail, it is possible by a careful analysis, in the case when all the boundaries are rigid, to discover the structure of the expansions generated. The expansions turn out to be simple power series in the critical parameter $\lambda = \sigma\beta\epsilon^{-\frac{1}{2}}$, where σ is the Prandtl number, β the thermal Rossby number and ϵ is the Taylor number. In view of the complicated nature of the flow such a simple result could hardly have been foreseen at the start, and indeed an analogous discussion of the problem in which the upper fluid surface is free soon runs into difficulties and is not pursued. The power series in λ cannot be expected to give useful representations of the various dependent variables when λ becomes of order unity, that is when conductive and convective effects are of equal significance in the transfer of heat. However, they show what the orders of magnitude of the various dependent variables are in the different regions of the flow when conduction and convection are of equal significance, and hence how it is possible to construct a self-consistent scheme for the description of the flow. In fact the determination of the flow field is reduced from a problem of solving a system of three equations, two of second order and one of fourth order, to one of solving a single equation of second order. One consequence of including the convective transfer of heat is that it becomes necessary for the side boundary layers to have a double structure even when the upper fluid surface is rigid, which is not necessary when there is no convection.

The single second-order equation referred to above is of course non-linear, and no full solution is found. Expansion in powers of λ linearizes this equation and allows the evaluation of approximate expressions for the various flow properties. Some results of this expansion are given, the most extensive being an expression for the Nusselt number which is evaluated through to the λ^4 term.

2. The governing equations

This analysis is performed under a number of simplifying assumptions, one of the most important being the Boussinesq approximation. It is supposed that the fluid is incompressible, that density variations are small, that they are linearly dependent only on temperature variations and that they are dynamically significant only in the buoyancy force they produce. Then the equation of steady motion referred to axes rotating with the steady angular velocity $\mathbf{\Omega}$ of the annulus is

$$u'_j \frac{\partial u'_i}{\partial x'_j} + 2(\mathbf{\Omega} \times \mathbf{u}')_i = \alpha g(T' - T_0)\,\delta_{i3} - \frac{1}{\rho_0}\frac{\partial p'}{\partial x'_i} + \nu \frac{\partial^2 u'_i}{\partial x'_j \partial x'_j}. \tag{1}$$

Here $\mathbf{u}'$ is fluid velocity measured relative to the rotating frame, $\mathbf{x}'$ is a Cartesian position vector, the 03 axis is vertically upwards, g is the acceleration due to gravity and ν is the kinematic viscosity which will be supposed constant. Also α is the coefficient of thermal expansion, ρ_0 is the fluid density corresponding to temperature T_0, T' is the local temperature and p' measures the departure of the pressure from the hydrostatic pressure that prevails when the fluid is at rest at a uniform temperature T_0. The assumption that centrifugal forces are negligible is also made.

We next introduce non-dimensional variables. These can be defined in terms of quantities already introduced, the distance ℓ between the outer and inner radii of the annulus and the difference ΔT between the temperatures applied at the two vertical walls. For definiteness, we shall suppose the inner wall to be held at constant temperature T_0 and the outer wall at $T_0 + \Delta T$. The following unprimed non-dimensional variables can then be defined

$$\mathbf{x} = \frac{\mathbf{x}'}{\ell}, \quad T = \frac{T' - T_0}{\Delta T}, \quad \mathbf{u} = \frac{2\Omega \mathbf{u}'}{\alpha g \Delta T}, \quad p = \frac{p'}{\alpha g \rho_0 \ell \, \Delta T}. \tag{2}$$

This scaling is suited to a flow in which the primary balance outside the boundary layers is that between the Coriolis force, the buoyancy force and the pressure gradient. In terms of these dimensionless variables, the equation of motion is

$$\beta(\mathbf{u}.\mathrm{grad})\mathbf{u} + (\mathbf{k} \times \mathbf{u}) = T\mathbf{k} - \mathrm{grad}\, p + \epsilon \nabla^2 \mathbf{u}, \tag{3}$$

where $\mathbf{k}$ is a unit vector in the upward vertical direction, and we have introduced the dimensionless parameters

$$\beta, \text{ the thermal Rossby number } = \alpha g \Delta T/(4\ell\Omega^2),$$

$$\epsilon, \text{ the Taylor number } = \nu/(2\Omega\ell^2).$$

In addition to the equation of motion we need the continuity and heat-transfer equations which, in terms of our dimensionless variables, are

$$\mathrm{div}\, \mathbf{u} = 0, \tag{4}$$

$$\sigma\beta \mathbf{u}.\mathrm{grad}\, T = \epsilon \nabla^2 T. \tag{5}$$

Here σ is the dimensionless Prandtl number ν/κ, where κ is the thermometric conductivity.

We shall be concerned with solving the equations under the assumptions that both β and ϵ are small. We therefore neglect the convective acceleration term in the equation of motion which is multiplied by β, but not of course the viscous term with factor ϵ which we can expect to be significant in the boundary layers. We shall also at first neglect the convective term in our heat-transfer equation. For this to be justified it is clear that the sizes of the small quantities $\sigma\beta$ and ϵ must be suitably related. It turns out that the approximation is justified provided $\sigma\beta\epsilon^{-\frac{1}{2}}$ is small. That this is the relevant condition rather than the apparent one that $\sigma\beta\epsilon^{-1}$ be small is due to the anistropy of the induced velocity field, and is just one illustration of the fact that the structure of an enclosed flow such as we are studying is not readily deducible *ab initio*, but becomes apparent only after the solution has been obtained.

Following Robinson, we make the further simplifying assumption that the radius of the annulus is much greater than its width or depth. This allows us to neglect curvature effects and work with Cartesian axes in a cross-section of the annulus. (Though convenient, this simplification is not essential and the problem could be worked without it. The methods remain the same but various expressions become more complicated.) We take the origin in the centre of the inner cooler side, the x-axis radially outwards, the y-axis along the length of the annulus and the z-axis vertically upwards. The flow, which is independent of y through the assumption of axial symmetry, takes place in the region bounded by the planes

$$x = 0, \quad x = 1, \quad z = \pm d/2\ell = \pm\gamma,$$

where d is the depth of the fluid, and γ is the geometrical parameter defined by the ratio $d/2\ell$. (Robinson considered the case of an annulus of square cross-section only for which $\gamma = \frac{1}{2}$.)

The temperature T satisfies the boundary conditions

$$T = 0 \quad \text{at} \quad x = 0, \quad T = 1 \quad \text{at} \quad x = 1, \quad \partial T/\partial z = 0 \quad \text{at} \quad z = \pm\gamma,$$

since the side walls are held at fixed temperature, while the upper and lower surfaces are insulating. The solution of the heat conduction equation $\nabla^2 T = 0$ subject to these conditions is

$$T = x, \tag{6}$$

so that temperature varies linearly with distance across the annulus.

Because of the axisymmetry and the corresponding lack of y-dependence, the continuity equation can be satisfied identically by means of the stream function ψ where

$$u = \partial\psi/\partial z, \quad w = -\partial\psi/\partial x. \tag{7}$$

Using this stream function, the second component of the equation of motion (3) becomes

$$\partial\psi/\partial z = \epsilon\nabla^2 v. \tag{8}$$

The pressure p can be eliminated between the first and third components of (3) to give, using our solution for T,

$$\epsilon\nabla^4\psi + \partial v/\partial z = 1. \tag{9}$$

We therefore have a pair of coupled equations for v and ψ as functions of x and z, and the next two sections will be concerned with their solution. Here and subsequently ∇^2 is simply the two dimensional Laplacian $\partial^2/\partial x^2 + \partial^2/\partial z^2$.

3. The flow in an annulus with four rigid walls

If all the boundaries are supposed rigid, the fluid velocity must vanish at each and we have the boundary conditions

$$\left.\begin{aligned} v = \psi = \partial\psi/\partial z = 0 \quad \text{on} \quad z = \pm\gamma, \\ v = \psi = \partial\psi/\partial x = 0 \quad \text{on} \quad x = 0, 1, \end{aligned}\right\} \tag{10}$$

for (8) and (9). Since we are interested only in the case of ϵ small, it is natural to tackle these equations by boundary-layer methods. The introduction of these can be delayed a little, as it is apparent that the governing equations (8) and (9) and the boundary conditions on the horizontal surfaces $z = \pm\gamma$ can be satisfied by a solution of the restricted form

$$v = V(z), \quad \psi = \epsilon^{\frac{1}{2}} F(z). \tag{11}$$

The functions V and F are found as the solutions of a coupled pair of ordinary differential equations and, neglecting terms which are small and of order $\exp(\epsilon^{-\frac{1}{2}})$, they are

$$F(z) = \gamma 2^{-\frac{1}{2}}\{1 - (\cos\zeta_1 - \sin\zeta_1)\, e^{\zeta_1} - (\cos\zeta_2 + \sin\zeta_2)\, e^{-\zeta_2}\}, \tag{12}$$

$$V(z) = z - \gamma e^{\zeta_1}\cos\gamma_1 + \gamma e^{-\zeta_2}\cos\zeta_2, \tag{13}$$

where
$$\zeta_1 = \frac{z-\gamma}{(2\epsilon)^{\frac{1}{2}}}, \quad \zeta_2 = \frac{z+\gamma}{(2\epsilon)^{\frac{1}{2}}}.$$

These expressions describe a flow which is a simple vertically sheared zonal flow away from the top and bottom, and has Ekman layers near these boundaries to adjust it to the required boundary conditions. In the Ekman layers, but not in the interior, there is motion in a cross-section of the annulus, consisting of a net horizontal flux of magnitude $\gamma(\frac{1}{2}\epsilon)^{\frac{1}{2}}$ in the positive x-direction in the bottom layer, and an equal and opposite flow in the top layer. When boundary-layer methods are used, the flows in the interior and Ekman layers are determined fully only after they are all matched.

We now introduce auxiliary functions ψ^* and v^* by setting
$$\psi = \epsilon^{\frac{1}{2}}F(z) + \psi^*, \quad v = V(z) + v^*.$$

These functions satisfy the homogeneous equations
$$\partial\psi^*/\partial z = \epsilon\nabla^2 v^*, \tag{14}$$
$$\epsilon\nabla^4\psi^* + \partial v^*/\partial z = 0, \tag{15}$$
and the boundary conditions
$$\left.\begin{array}{l} \psi^* = \partial\psi^*/\partial z = v^* = 0 \quad \text{on} \quad z = \pm\gamma, \\ \partial\psi^*/\partial x = 0, \quad v^* = -V(z), \quad \psi^* = -\epsilon^{\frac{1}{2}}F(z) \quad \text{on} \quad x = 0, \quad x = 1. \end{array}\right\} \tag{16}$$

The auxiliary functions have non-zero boundary conditions at the side walls only, and can differ significantly from zero near these walls only.

There are two methods for solving the equations for v^* and ψ^*. One is to find separable solutions of the equations, and then take a suitable combination of these separable solutions. This is the method used by Stewartson in his analysis of the present equations in a different context. One gets a complicated transcendental equation for the separation parameter, which can be solved approximately to the lowest order when ϵ is small, but the working becomes very messy when higher degrees of approximation, which Stewartson did not investigate, are required. The analysis of separable solutions satisfying the requisite zero boundary conditions on the top and bottom surfaces show that all the solutions decay exponentially to zero away from the walls in distances $O(\epsilon^{\frac{1}{3}})$, apart from a single exceptional solution which decays instead in a distance $O(\epsilon^{\frac{1}{4}})$. The exceptional solution arises in what Stewartson terms the 'symmetric case' but not in the 'antisymmetric case'. The symmetric and antisymmetric cases are respectively those in which the solution for v^* is a symmetric and antisymmetric function of z when the top and bottom boundaries are taken, as in the present work, at equal and opposite values of z. Stewartson's terminology was suited to his problem where he was principally concerned with the function v^*. However, the terminology is not generally appropriate as it is easily seen from the governing equations that if v^* is a symmetric function of z, ψ^* is antisymmetric in z, and vice versa. The solution we require to satisfy the boundary conditions (16) needs to have v^* antisymmetric and ψ^* symmetric in z, so that the exceptional separable solution which gives a side boundary layer of width $O(\epsilon^{\frac{1}{4}})$ does not arise.

We shall proceed to solve (14) and (15) by boundary-layer methods following Greenspan & Howard (1963). The form of the solutions for ψ^* and v^* near $x = 0$ only will be given, as that near the other wall $x = 1$ has a precisely similar structure, and can be obtained from that near $x = 0$ via the substitution of $(1 - x)$ for x throughout. For the side boundary-layer analysis, we need to introduce a scaled co-ordinate $\xi = x\epsilon^{-\frac{1}{3}}$. (14) and (15) then have the approximate form

$$\frac{\partial \psi^*}{\partial z} = \epsilon^{\frac{1}{3}} \frac{\partial^2 v^*}{\partial \xi^2}, \quad \frac{\partial^4 \psi^*}{\partial \xi^4} + \epsilon^{\frac{1}{3}} \frac{\partial v^*}{\partial z} = 0, \tag{17}$$

with terms $O(\epsilon^{\frac{2}{3}})$ compared with those retained being neglected. These approximate equations cease to be valid within distances $O(\epsilon^{\frac{1}{2}})$ from the top and bottom surfaces. Equations (17) must be solved subject to the boundary conditions (16) which, away from the top and bottom, become

$$\partial \psi^*/\partial \xi = 0, \quad v^* = -z, \quad \psi^* = -\gamma(\tfrac{1}{2}\epsilon)^{\frac{1}{2}} \quad \text{on} \quad \xi = 0. \tag{18}$$

A separate analysis is needed for the parts of the side layers within $O(\epsilon^{\frac{1}{2}})$ of the top and bottom. For these regions, we use the scaled z-co-ordinates ζ_1 and ζ_2 defined above. The flow equations have the approximate form

$$\left.\begin{aligned} \partial \psi^*/\partial \zeta &= (\tfrac{1}{2}\epsilon)^{\frac{1}{2}}\, \partial^2 v^*/\partial \zeta^2 + O(\epsilon^{\frac{5}{6}}), \\ \partial^4 \psi^*/\partial \zeta^4 + 2(2\epsilon)^{\frac{1}{2}}\, \partial v^*/\partial \zeta + O(\epsilon^{\frac{5}{6}}) &= 0, \end{aligned}\right\} \tag{19}$$

with ζ either of ζ_1 or ζ_2. Throughout the side layer, v^* and ψ^* must tend to zero as $\xi \to \infty$ to match onto the interior. Also, the solutions in the different sections of the side layer must match across their common boundaries, and this matching must be carried out in order to determine fully the solution for the side layer. It should be noted that the flow in the Ekman extensions at the top and bottom of the side layer does not satisfy the side-wall boundary conditions. The reason for this is that its range of validity does not extend to the side wall. There are two square corner regions of linear dimension $O(\epsilon^{\frac{1}{2}})$ in which yet another scaling of the equations is relevant. This scaling is one which does not allow any terms to be neglected but, as Greenspan & Howard found, the remainder of the flow can be solved without investigating these regions.

The analysis of the Ekman extensions of the side boundary layers is straightforward, as the differential equations (19) which govern it are essentially ordinary differential equations. Near $z = \gamma$, we have the solution

$$v^* = A(\xi)\{1 - e^{\zeta_1}\cos\zeta_1\}, \quad \psi^* = A(\xi)\,(\tfrac{1}{2}\epsilon)^{\frac{1}{2}}\{1 - (\cos\zeta_1 - \sin\zeta_1)\,e^{\zeta_1}\}, \tag{20}$$

and near $z = -\gamma$,

$$v^* = B(\xi)\{-1 + e^{-\zeta_2}\cos\zeta_2\}, \quad \psi^* = B(\xi)\,(\tfrac{1}{2}\epsilon)^{\frac{1}{2}}\{1 - (\cos\zeta_2 + \sin\zeta_2)\,e^{-\zeta_2}\}. \tag{21}$$

Here $A(\xi)$ and $B(\xi)$ are as yet unknown. They must be such that $A(\infty) = B(\infty) = 0$, and must also match onto the main part of the side boundary layer. Since the solution in this region is symmetric in v^* and antisymmetric in ψ^* as functions of z, it is necessary that

$$A(\xi) = B(\xi).$$

It is convenient to expand

$$\left.\begin{aligned} \psi^*(\xi,z) &= \epsilon^{\frac{1}{3}}\psi_0+\epsilon^{\frac{1}{2}}\psi_1+O(\epsilon^{\frac{2}{3}}),\\ v^*(\xi,z) &= v_0+\epsilon^{\frac{1}{6}}v_1+O(\epsilon^{\frac{1}{3}}),\end{aligned}\right\} \tag{22}$$

for the investigation of the main side boundary layer. To match the side layer onto its extensions (20) and (21), we require $v^*\to\pm A(\xi)$, $\psi^*\to A(\xi)(\frac{1}{2}\epsilon)^{\frac{1}{2}}$ as $z\to\pm\gamma$, so we also expand

$$A(\xi) = A_0(\xi)+\epsilon^{\frac{1}{6}}A_1(\xi)+O(\epsilon^{\frac{1}{3}}),$$

and get the conditions

$$\psi_0\to 0,\quad v_0\to\pm A_0(\xi),\quad \psi_1\to A_0(\xi)/2^{\frac{1}{2}}\quad \text{as}\quad z\to\pm\gamma.$$

The function ψ_0 is next expanded in terms of the complete set of functions $\sin m\pi(z+\gamma)/2\gamma$, where m is any integer. Because of the symmetry of ψ_0 as a function of z, odd values of m only are required, so that

$$\psi_0 = \sum_{n=0}^{\infty} f_{0,n}(\xi)\sin\frac{(2n+1)\pi(z+\gamma)}{2\gamma}. \tag{23}$$

This sine series can be differentiated term by term since ψ_0 vanishes at $z=\pm\gamma$, and the resulting cosine series can also be differentiated term by term (Jeffreys & Jeffreys, 1956). It follows from (17) by eliminating v^* that

$$\partial^6\psi^*/\partial\xi^6+\partial^2\psi^*/\partial z^2 = 0, \tag{24}$$

and so the functions $f_{0,n}(\xi)$ must satisfy

$$d^6f_{0,n}/d\xi^6 = (2n+1)^2\pi^2 f_{0,n}/4\gamma^2. \tag{25}$$

It also follows that

$$\frac{\partial v_0}{\partial z} = -\frac{\partial^4\psi_0}{\partial\xi^4} = -\sum_{n=0}^{\infty} f_{0,n}^{(\text{iv})}(\xi)\sin\frac{(2n+1)\pi(z+\gamma)}{2\gamma}. \tag{26}$$

Integrating with respect to z, and using the fact that v^* is antisymmetric in z, we get

$$v_0 = \sum_{n=0}^{\infty}\frac{2\gamma f_{0,n}^{(\text{iv})}(\xi)}{(2n+1)\pi}\cos\frac{(2n+1)\pi(z+\gamma)}{2\gamma}. \tag{27}$$

The boundary conditions (18) at the wall $\xi=0$ must now be applied, and they require that

$$f_{0,n}(0) = f'_{0,n}(0) = 0,\quad f_{0,n}^{(\text{iv})}(0) = 4/(2n+1)\pi. \tag{28}$$

The relevant solution of (25) that satisfies these boundary conditions and tends to zero as $\xi\to\infty$ is

$$f_{0,n}(\xi) = \frac{1}{\gamma\omega_n^7}\left\{e^{-\omega_n\xi}-\frac{2e^{-\frac{1}{2}\omega_n\xi}}{\sqrt{3}}\cos\left(\frac{\omega_n\xi\sqrt{3}}{2}+\frac{\pi}{6}\right)\right\}, \tag{29}$$

with the abbreviation $\omega_n = [(2n+1)\pi/2\gamma]^{\frac{1}{3}}$. It follows that

$$f_{0,n}^{(\text{iv})}(\xi) = \frac{1}{\gamma\omega_n^3}\left\{e^{-\omega_n\xi}+\frac{2e^{-\frac{1}{2}\omega_n\xi}}{\sqrt{3}}\cos\left(\frac{\omega_n\xi\sqrt{3}}{2}-\frac{\pi}{6}\right)\right\}. \tag{30}$$

The matching onto the Ekman extensions is accomplished if we take

$$A_0(\xi) = -\sum_{n=0}^{\infty} \frac{f_{0,n}^{(\mathrm{iv})}(\xi)}{\omega_n^3}, \tag{31}$$

for the expressions for v then match, and the determination of the leading terms in the expressions for the flow variables in all the regions is now complete.

Since ψ_0 vanishes both at the wall $\xi = 0$ and at the outer edge $\xi = \infty$ of the side layer, it represents a closed cell circulation in the (x, z)-plane with flux $O(\epsilon^{\frac{1}{3}})$. A net downward flux of magnitude $O(\epsilon^{\frac{1}{2}})$ only is needed to link up the fluxes in the Ekman layers determined earlier, and this is given by the next terms in the expansion.

The determination of the next terms is very similar to our previous working. Although ψ_1 does not vanish at $z = \pm\gamma$, $\psi_1 - A_0(\xi)/\sqrt{2}$ does and so we expand it in a sine series

$$\psi_1 = \frac{A_0(\xi)}{\sqrt{2}} + \sum_{n=0}^{\infty} f_{1,n}(\xi) \sin \frac{(2n+1)\pi(z+\gamma)}{2\gamma}, \tag{32}$$

and this too can be differentiated twice term by term with respect to z. Equation (24) for ψ^* requires that the functions $f_{1,n}(\xi)$ satisfy the equations

$$d^6 f_{1,n}/d\xi^6 - \omega_n^6 f_{1,n} = -A_0^{(\mathrm{vi})}(\xi)\sqrt{2}/\gamma\omega_n^3. \tag{33}$$

An expression for v_1 is found as before by integrating the second of equations (17) and using the antisymmetry condition and is

$$v_1 = -\frac{zA_0^{(\mathrm{iv})}(\xi)}{\sqrt{2}} + \sum_{n=0}^{\infty} \frac{f_{1,n}^{(\mathrm{iv})}(\xi)}{\omega_n^3} \cos \frac{(2n+1)\pi(z+\gamma)}{2\gamma}. \tag{34}$$

The boundary conditions at $\xi = 0$ are $v_1 = \partial\psi_1/\partial\xi = 0$, $\psi_1 = -\gamma/\sqrt{2}$, and so require

$$f_{1,n}(0) = 0, \quad \text{since} \quad A_0(0) = -\sum_{n=0}^{\infty} \frac{8\gamma}{(2n+1)^2\pi^2} = -\gamma,$$

$$f'_{1,n}(0) = -\frac{A'_0(0)\sqrt{2}}{\gamma\omega_n^3}, \quad f_{1,n}^{(\mathrm{iv})}(0) = -\frac{A_0^{(\mathrm{iv})}(0)\sqrt{2}}{\gamma\omega_n^3}.$$

The solution of (33) that satisfies these boundary conditions and tends to zero as $\xi \to \infty$ is

$$f_{1,n}(\xi) = \frac{A'_0(0)}{\gamma\omega_n^4\sqrt{2}} \left\{ e^{-\omega_n\xi} - 2e^{-\frac{1}{2}\omega_n\xi} \cos\left(\frac{\omega_n\xi\sqrt{3}}{2} - \frac{\pi}{3}\right) \right\}$$
$$- \frac{A_0^{(\mathrm{iv})}(0)}{\gamma\omega_n^7\sqrt{2}} \left\{ e^{-\omega_n\xi} - \frac{2}{\sqrt{3}} e^{-\frac{1}{2}\omega_n\xi} \cos\left(\frac{\omega_n\xi\sqrt{3}}{2} + \frac{\pi}{6}\right) \right\} + \mathscr{L}\left(\omega_n, -\frac{A_0^{(\mathrm{vi})}(\xi)\sqrt{2}}{\gamma\omega_n^3}\right). \tag{35}$$

For the last term in this expression, we have introduced the notation $\mathscr{L}(\omega, f)$ for the solution of the sixth-order inhomogeneous equation

$$d^6y/d\xi^6 - \omega^6 y = f(\xi)$$

which $\to 0$ as $\xi \to \infty$, it being supposed that $f \to 0$ exponentially as $\xi \to \infty$, and which satisfies the boundary conditions

$$y = dy/d\xi = d^4y/d\xi^4 = 0 \quad \text{at} \quad \xi = 0.$$

It can be shown that

$$6\omega^5\mathscr{L}(\omega,f) = e^{\omega\xi}\int_{\infty}^{\xi} f(\eta)\,e^{-\omega\eta}\,d\eta - e^{-\omega\xi}\int_{0}^{\xi} f(\eta)\,e^{\omega\eta}\,d\eta$$

$$+2e^{-\omega\xi}\int_{0}^{\infty} f(\eta)\,e^{-\frac{1}{2}\omega\eta}\cos\left(\frac{\omega\eta\sqrt{3}}{2}+\frac{\pi}{3}\right)d\eta + 2e^{-\frac{1}{2}\omega\xi}\cos\left(\frac{\omega\xi\sqrt{3}}{2}-\frac{\pi}{3}\right)\int_{0}^{\infty} f(\eta)\,e^{-\omega\eta}\,d\eta$$

$$+4e^{-\frac{1}{2}\omega\xi}\cos\left(\frac{\omega\xi\sqrt{3}}{2}-\frac{\pi}{6}\right)\int_{0}^{\infty} f(\eta)\,e^{-\frac{1}{2}\omega\eta}\sin\frac{\omega\eta\sqrt{3}}{2}\,d\eta$$

$$+2e^{\frac{1}{2}\omega\xi}\int_{\infty}^{\xi} f(\eta)\,e^{-\frac{1}{2}\omega\eta}\cos\left[\frac{\pi}{3}+\frac{\omega(\xi-\eta)\sqrt{3}}{2}\right]d\eta$$

$$-2e^{-\frac{1}{2}\omega\xi}\int_{0}^{\xi} f(\eta)\,e^{\frac{1}{2}\omega\eta}\cos\left[\frac{\pi}{3}+\frac{\omega(\eta-\xi)\sqrt{3}}{2}\right]d\eta. \tag{36}$$

The value of $A_1(\xi)$ is the limiting value of v_1 as $z \to \gamma$ and so

$$A_1(\xi) = -\frac{\gamma A_0^{(\mathrm{iv})}(\xi)}{\sqrt{2}} - \sum_{n=0}^{\infty}\frac{f_{1,n}^{(\mathrm{iv})}(\xi)}{\omega_n^3},$$

and the second terms in our expansions are fully determined. Terms in higher powers of ϵ can clearly be determined iteratively in an obvious way, but we shall not continue the expansion. Since the function ψ_1 vanishes at the outer edge of the side layer, but equals $-\gamma/\sqrt{2}$ at the side wall, we have the net downward flux of $\gamma\sqrt{(\frac{1}{2}\epsilon)}$ in the side layer required to complete the meridional circulation through the Ekman layers. This current is turned round from flowing horizontally to flowing vertically and vice versa in the Ekman extensions of the side layer.

The boundary-layer analysis just given differs from Robinson's in certain significant respects. Robinson omits consideration of the top and bottom extensions of the side boundary layer and determines the flow in the side layer simply by solving (17) subject to the boundary conditions (18). In terms of the present notation, which is generally the same as Robinson's though the boundaries are placed differently, Robinson's method is to look for a series solution $\psi^* = \sum_n f_n(\xi)\sin\omega_n^3(z+\gamma)$, since $\psi^* = 0$ on $z = \pm\gamma$. The f_n are determined by choosing suitable solutions of the homogeneous equation $d^6f_n/d\xi^6 = \omega_n^6 f_n$ which $\to 0$ as $\xi \to \infty$ and allow the right boundary conditions to be satisfied on $\xi = 0$. As the above discussion has shown, this treatment is inadequate. One can not apply the boundary conditions at $z = \pm\gamma$ directly onto the side boundary layer. This layer must be matched onto the boundaries $z = \pm\gamma$ through Ekman type boundary layers at its top and bottom. Robinson is fortunate in that, as we have seen, this matching requires the leading term ψ_0 of ψ^*, though not ψ_1, to vanish as $z \to \pm\gamma$ in the side boundary layers, so that his treatment should give ψ_0 and v_0 correctly. His quoted results for these leading terms are in error, however. Though his expressions for them both satisfy the sixth-order equation (24) and the correct boundary conditions, they do not satisfy the correct inter-relations (17).

4. The flow when the upper surface is free

Since much of the experimental work on annular flows is done with the upper surface of the fluid free, we shall now rework the previous problem for this case. The new boundary conditions on the upper surface are

$$\psi = \partial v/\partial z = \partial u/\partial z = \partial^2\psi/\partial z^2 = 0 \quad \text{on} \quad z = \gamma. \tag{37}$$

(We have supposed that the free upper surface remains plane. The magnitude of vertical surface displacements relative to the width of the annulus is of the order of β times the ratio of centrifugal force to gravity, and we neglect this in line with our previous approximations.)

The method of dealing with the present problem follows closely that of the previous section though there are some new features. Once again the equations and boundary conditions on top and bottom can be satisfied by functions of z only. After solving for these we write as before

$$\left.\begin{aligned} v &= (z+\gamma)-(2\epsilon)^{\frac{1}{2}}-(\tfrac{1}{2}\epsilon)^{\frac{1}{2}}(\cos\zeta_1+\sin\zeta_1)\,e^{\zeta_1}+(2\epsilon)^{\frac{1}{2}}e^{-\zeta_2}\cos\zeta_2+v^*,\\ \psi &= \epsilon\{1-e^{\zeta_1}\cos\zeta_1-(\cos\zeta_2+\sin\zeta_2)\,e^{-\zeta_2}\}+\psi^*, \end{aligned}\right\} \tag{38}$$

and then go on to solve for the functions v^* and ψ^* which differ significantly from zero near the side walls only. The interior flow again consists of a vertically sheared zonal flow, though this is given by a different expression. A more significant difference is that the meridional flux through the Ekman layers, of magnitude ϵ in the negative x-direction in the top layer and in the positive x-direction in the bottom layer, is weaker by $O(\epsilon^{\frac{1}{2}})$ from what it was previously. This result is in line with that of Hide (1964) who found that the Ekman boundary layer associated with a free surface is weaker than that associated with a rigid surface.

The functions ψ^* and v^* satisfy the same homogeneous equations (14) and (15) as before, and the boundary conditions are now

$$\left.\begin{aligned} &\psi^* = \partial\psi^*/\partial z = v^* = 0 \quad \text{on} \quad z=-\gamma, \quad \psi^* = \partial^2\psi^*/\partial z^2 = \partial v^*/\partial z = 0 \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{on} \quad z=\gamma,\\ &\partial\psi^*/\partial x = 0, \quad v^* = -z-\gamma+(2\epsilon)^{\frac{1}{2}}, \quad \psi^* = -\epsilon \quad \text{on} \quad x=0, \quad x=1, \end{aligned}\right\} \tag{39}$$

except within distances $O(\epsilon^{\frac{1}{2}})$ of $z = \pm\gamma$.

The analysis of the side boundary layers is now more complicated. The functions ψ^* and v^* are no longer simply symmetric and antisymmetric functions of z respectively, and so, as Stewartson's separation of variables analysis shows, we must expect side boundary layers of double structure with thicknesses $O(\epsilon^{\frac{1}{3}})$ and $O(\epsilon^{\frac{1}{4}})$. Solutions for the flow must be derived for the inner side boundary layer of thickness $O(\epsilon^{\frac{1}{3}})$, the upper and lower extensions of this layer within distances $O(\epsilon^{\frac{1}{2}})$ from the top and bottom, the outer side boundary layer of thickness $O(\epsilon^{\frac{1}{4}})$, and its upper and lower extensions. The solutions for the outer side layer must be matched onto the interior flow and the boundary conditions on $x = 0$ must be applied to the inner side boundary layer, but not its extensions. Only when we have solved for all these regions and matched across all adjacent boundaries can the solution be fully determined. This problem is similar to that described in

§6 of the paper by Greenspan & Howard, though they have the additional complication of a time dependence.

The boundary conditions on the side $x = 0$ and the form of the inner side boundary-layer equations require that v^* is $O(1)$ and ψ^* is $O(\epsilon^{\frac{1}{3}})$ in this region as before. In the upper and lower extensions of this layer and throughout the outer side boundary layer, v^* must be $O(1)$ for matching, while the governing equations require ψ^* to be $O(\epsilon^{\frac{1}{2}})$ only. We shall start by solving for the outer layer. Introducing the suitably scaled variable $\eta = x\epsilon^{-\frac{1}{4}}$, the governing equations are

$$\frac{\partial \psi^*}{\partial z} = \epsilon^{\frac{1}{2}} \frac{\partial^2 v^*}{\partial \eta^2} + O(\epsilon), \quad \frac{\partial v^*}{\partial z} = O(\epsilon^{\frac{1}{2}}), \tag{40}$$

while for the upper and lower ends of this layer, they are

$$\frac{\partial \psi^*}{\partial \zeta} = (\tfrac{1}{2}\epsilon)^{\frac{1}{2}} \frac{\partial^2 v^*}{\partial \zeta^2} + O(\epsilon), \quad \frac{\partial^4 \psi^*}{\partial \zeta^4} + 2(2\epsilon)^{\frac{1}{2}} \frac{\partial v^*}{\partial \zeta} + O(\epsilon) = 0. \tag{41}$$

This last pair of equations differs from those for the previous problem (19) which are still relevant to the extensions of the inner side layer only in the magnitude of the errors.

The integrals of (40) for the outer side layer are

$$v^* = U(\eta) + O(\epsilon^{\frac{1}{2}}), \quad \psi^* = \epsilon^{\frac{1}{2}}[W(\eta) + zU''(\eta)] + O(\epsilon), \tag{42}$$

while those of (41) for its upper and lower extensions are respectively

$$v^* = A(\eta) + O(\epsilon^{\frac{1}{2}}), \qquad \psi^* = 0 + O(\epsilon), \tag{43}$$

$$\left.\begin{aligned} v^* &= B(\eta)\{1 - e^{-\zeta_2}\cos\zeta_2\} + O(\epsilon^{\frac{1}{2}}), \\ \psi^* &= -B(\eta)(\tfrac{1}{2}\epsilon)^{\frac{1}{2}}\{1 - (\cos\zeta_2 + \sin\zeta_2)e^{-\zeta_2}\} + O(\epsilon). \end{aligned}\right\} \tag{44}$$

The four functions A, B, U and W are $O(1)$ functions of integration. The matching of these solutions requires

$$U = A = B, \quad W + \gamma U'' = 0, \quad -B = 2^{\frac{1}{2}}(W - \gamma U''); \tag{45}$$

relations which can be solved to give

$$U(\eta) = C\exp\left(\frac{-\eta}{\gamma^{\frac{1}{2}}2^{\frac{3}{4}}}\right) + O(\epsilon^{\frac{1}{2}}), \quad W(\eta) = -\frac{C}{2\sqrt{2}}\exp\left(\frac{-\eta}{\gamma^{\frac{1}{2}}2^{\frac{3}{4}}}\right) + O(\epsilon^{\frac{1}{2}}). \tag{46}$$

Here C is a constant, and the condition of matching onto the interior flow with v^*, $\psi^* \to 0$ as $\eta \to \infty$ has been applied. The constant C can be determined only after we have matched onto the inner side boundary layer.

For the upper and lower extremities of the inner side layer the respective integrals are similar to (43) and (44) for the extensions of the outer side layer and can be written as

$$v^* = D(\xi) + O(\epsilon^{\frac{1}{3}}), \quad \psi^* = 0 + O(\epsilon^{\frac{5}{6}}), \tag{47}$$

$$\left.\begin{aligned} v^* &= E(\xi)\{1 - e^{-\zeta_2}\cos\zeta_2\} + O(\epsilon^{\frac{1}{3}}), \\ \psi^* &= -E(\xi)(\tfrac{1}{2}\epsilon)^{\frac{1}{2}}\{1 - (\cos\zeta_2 + \sin\zeta_2)e^{-\zeta_2}\} + O(\epsilon^{\frac{5}{6}}), \end{aligned}\right\} \tag{48}$$

where D and E are $O(1)$ functions of integration. The inner side layer proper requires the solution of equations

$$\epsilon^{\frac{1}{3}}\frac{\partial^2 v^*}{\partial \xi^2}+O(\epsilon)=\frac{\partial \psi^*}{\partial z},\quad \frac{\partial^4 \psi}{\partial \xi^4}+\epsilon^{\frac{1}{3}}\frac{\partial v^*}{\partial z}=O(\epsilon), \tag{49}$$

with ψ^* now being $O(\epsilon^{\frac{1}{3}})$.

It is convenient for the solution of the main inner side layer to expand functions in powers of ϵ. The series must now be in ascending powers of $\epsilon^{\frac{1}{12}}$, as this is the lowest common multiple of the powers $\epsilon^{\frac{1}{4}}$, $\epsilon^{\frac{1}{3}}$ and $\epsilon^{\frac{1}{2}}$ that arise in the problem. Thus

$$\left.\begin{aligned}
\psi^* &= \epsilon^{\frac{1}{3}}\psi_0+\epsilon^{\frac{5}{12}}\psi_1+\epsilon^{\frac{1}{2}}\psi_2+\epsilon^{\frac{7}{12}}\psi_3+\dots,\\
v^* &= v_0+\epsilon^{\frac{1}{12}}v_1+\epsilon^{\frac{1}{6}}v_2+\epsilon^{\frac{1}{4}}v_3+\dots,\\
D &= D_0+\epsilon^{\frac{1}{12}}D_1+\epsilon^{\frac{1}{6}}D_2+\epsilon^{\frac{1}{4}}D_3+\dots,\\
E &= E_0+\epsilon^{\frac{1}{12}}E_1+\epsilon^{\frac{1}{6}}E_2+\epsilon^{\frac{1}{4}}E_3+\dots,\\
C &= C_0+\epsilon^{\frac{1}{12}}C_1+\epsilon^{\frac{1}{6}}C_2+\epsilon^{\frac{1}{4}}C_3+\dots.
\end{aligned}\right\} \tag{50}$$

The fact that the series solutions (50) of equations (49) must match onto solutions (47) and (48) requires that both ψ_0 and $\psi_1 \to 0$ as $z \to \pm\gamma$, so we expand

$$\left.\begin{aligned}
\psi_0+\epsilon^{\frac{1}{12}}\psi_1 &= \sum_{m=1}^{\infty}\{f_{0,m}(\xi)+\epsilon^{\frac{1}{12}}f_{1,m}(\xi)\}\sin\sigma_m^3(z+\gamma),\\
\sigma_m &= (m\pi/2\gamma)^{\frac{1}{3}},
\end{aligned}\right\} \tag{51}$$

and can differentiate this series twice term by term with respect to z. The functions $f_{0,m}$ and $f_{1,m}$ must therefore both be solutions of the homogeneous equation

$$d^6y/d\xi^6=\sigma_m^6 y. \tag{52}$$

It follows then from (49) that

$$v_0+\epsilon^{\frac{1}{12}}v_1=\alpha+\beta\xi+\sum_{m=1}^{\infty}\frac{[f_{0,m}^{(\text{iv})}(\xi)+\epsilon^{\frac{1}{12}}f_{1,m}^{(\text{iv})}(\xi)]\cos\sigma_m^3(z+\gamma)}{\sigma_m^3}, \tag{53}$$

where α and β are constants of integration. The side boundary conditions must next be applied at $\xi=0$. They give $f_{0,m}(0)=f_{1,m}(0)=f'_{0,m}(0)=f'_{1,m}(0)=0$, and, expanding $-(z+\gamma)$ in terms of the functions $\cos\sigma_m^3(z+\gamma)$, $m=0$ to ∞, which are complete in $-\gamma\leqslant z\leqslant\gamma$,

$$\alpha=-\gamma,\quad f_{0,m}^{(\text{iv})}(0)=\frac{1-\cos m\pi}{\gamma\sigma_m^3},\quad f_{1,m}^{(\text{iv})}(0)=0. \tag{54}$$

It follows that $f_{1,m}(\xi)=0$, and that ψ_0 is identical to what it is in the problem of §3. Note that although ψ^* and v^* do not necessarily $\to 0$ in the inner layer as $\xi\to\infty$, the requirement of matching onto the outer layer solution precludes exponential growth. Finally, the matching of the zonal velocity (53) as $\xi\to\infty$ with the solution for the outer layer gives

$$C_0=-\gamma,\quad C_1=0,\quad \beta=\gamma^{\frac{1}{2}}\epsilon^{\frac{1}{12}}/2^{\frac{1}{4}}. \tag{55}$$

The first two significant terms in the expansion of the solution in powers of $\epsilon^{\frac{1}{12}}$ are now determined in all regions. For the inner side layer

$$\left.\begin{aligned} \psi_0 &= \sum_{n=0}^{\infty} \frac{\sin\omega_n^3(z+\gamma)}{\gamma\omega_n^7}\left\{e^{-\omega_n\zeta}-\frac{2e^{-\frac{1}{2}\omega_n\zeta}}{\sqrt{3}}\cos\left(\frac{\omega_n\xi\sqrt{3}}{2}+\frac{\pi}{6}\right)\right\}, \quad \psi_1 = 0, \\ v_0 &= -\gamma+\sum_{n=0}^{\infty}\frac{\cos\omega_n^3(z+\gamma)}{\gamma\omega_n^6}\left\{e^{-\omega_n\xi}+\frac{2e^{-\frac{1}{2}\omega_n\xi}}{\sqrt{3}}\cos\left(\frac{\omega_n\xi\sqrt{3}}{2}-\frac{\pi}{6}\right)\right\}, \quad v_1 = \frac{\gamma^{\frac{1}{2}}\xi}{2^{\frac{3}{4}}}, \end{aligned}\right\} \tag{56}$$

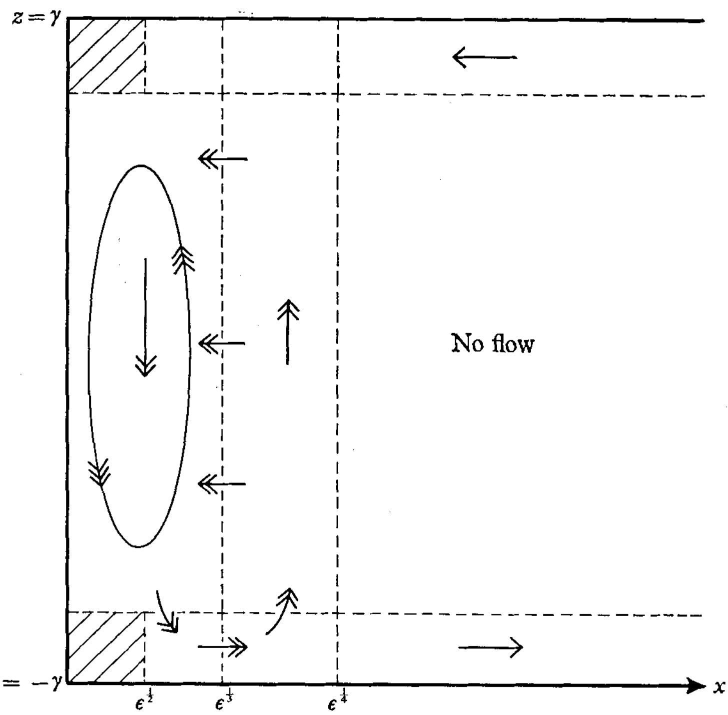

FIGURE 1. The meridional flow in the (x, z)-plane when the upper surface is free. The dashed lines separate the various boundary-layer regions, and the arrows indicate fluxes of different strengths. The triple-headed arrows are for $O(\epsilon^{\frac{1}{3}})$ fluxes, the double-headed arrows for $O(\epsilon^{\frac{1}{2}})$ fluxes and the single-headed arrows for $O(\epsilon)$ fluxes.

and the requirement of matching v^* onto the Ekman extensions as $z \to \pm\gamma$ determines

$$\left.\begin{aligned} D_0(\xi) &= -\gamma-\sum_{n=0}^{\infty}\frac{1}{\gamma\omega_n^6}\left\{e^{-\omega_n\xi}+\frac{2e^{-\frac{1}{2}\omega_n\xi}}{\sqrt{3}}\cos\left(\frac{\omega_n\xi\sqrt{3}}{2}-\frac{\pi}{6}\right)\right\}, \\ E_0(\xi) &= -2\gamma - D_0(\xi), \quad D_1(\xi) = E_1(\xi) = \gamma^{\frac{1}{2}}\xi/2^{\frac{3}{4}}. \end{aligned}\right\} \tag{57}$$

The matching of the extensions of the inner and outer side layers onto each other is a by-product of the matching of the layers themselves.

In the present problem, just as in the previous one, the largest component of the meridional flow is a closed circulation of magnitude $O(\epsilon^{\frac{1}{3}})$ in the inner side layer. Though we have not yet discussed the flow to $O(\epsilon^{\frac{1}{2}})$ for the inner side layer, we can deduce that it must consist of a net downflow of magnitude $\gamma(\frac{1}{2}\epsilon)^{\frac{1}{2}}$, since the solutions we have already determined show that there is a downflow of this magnitude from the inner side layer into its lower extension, across the common boundary of this extension and the lower extension of the outer layer, up into the main outer layer and then back into the inner side layer (see figure 1). Thus to $O(\epsilon^{\frac{1}{2}})$ we again have a closed circulation that is confined to the side. We shall not get anything other than this until we get to the $O(\epsilon)$ through-flow needed to link up the flows through the main Ekman layers.

The zonal velocity v around the annulus grows from its boundary value of zero at $x = 0$ to its value for the interior flow in two stages. It is clear from the series (53) we derived for v and the subsequent analysis that the outer side layer is needed to bring v up to its interior value $v(x = 0, z)$ if this interior value is such that

$$\int_{-\gamma}^{\gamma} v(x=0)\,dz \neq 0.$$

This is now the case. As far as the $O(1)$ terms are concerned, v grows to the value z across the inner side layer but the outer side layer is needed to bring v up to its interior value of $(z+\gamma)$.

The terms with subscripts 2 and 3 give a detailed description of the $O(\epsilon^{\frac{1}{2}})$ meridional flow in the inner side layer. Since it is necessary that $(\psi_2 + \epsilon^{\frac{1}{12}}\psi_3) \to 0$ as $z \to \gamma$ and $\to -(E_0(\xi) + \epsilon^{\frac{1}{12}}E_1(\xi))/\sqrt{2}$ as $z \to -\gamma$, it is convenient to expand

$$\psi_2 + \epsilon^{\frac{1}{12}}\psi_3 - \frac{(z-\gamma)\,[E_0 + \epsilon^{\frac{1}{12}}E_1]}{2\gamma\sqrt{2}} = \sum_{m=1}^{\infty} [f_{2,m}(\xi) + \epsilon^{\frac{1}{12}}f_{3,m}(\xi)] \sin\sigma_m^3(z+\gamma). \tag{58}$$

The expression on the left vanishes at both $z = \pm\gamma$, so that the series on the right may be differentiated twice term by term with respect to z. Equations (49) for ψ^* and v^* require

$$\left(\frac{d^6}{d\xi^6} - \sigma_m^6\right)[f_{2,m}(\xi) + \epsilon^{\frac{1}{12}}f_{3,m}(\xi)] = \frac{E_0^{(\mathrm{vi})}(\xi)}{\gamma\sigma_m^3\sqrt{2}}, \tag{59}$$

$$\begin{aligned} v_2 + \epsilon^{\frac{1}{12}}v_3 = a + b\xi + \frac{1}{2\gamma\sqrt{2}}\int_0^{\xi} d\mu \int_0^{\mu} E_0(\lambda)\,d\lambda + \frac{\epsilon^{\frac{1}{12}}\xi^3}{24\gamma^{\frac{1}{2}}2^{\frac{1}{4}}} \\ + \frac{1}{2\gamma\sqrt{2}}\left\{\frac{2\gamma^2}{3} - \frac{(z-\gamma)^2}{2}\right\}E_0^{(\mathrm{iv})}(\xi) + \sum_{m=1}^{\infty}\frac{[f_{2,m}^{(\mathrm{iv})}(\xi) + \epsilon^{\frac{1}{12}}f_{3,m}^{(\mathrm{iv})}(\xi)]\cos\sigma_m^3(z+\gamma)}{\sigma_m^3}, \end{aligned} \tag{60}$$

with a and b constants as yet undetermined. The boundary conditions at the side $\xi = 0$ require

$$\begin{aligned} a = f_{2,m}(0) = f_{3,m}(0) = f_{3,m}^{(\mathrm{iv})}(0) = 0, \\ f'_{2,m}(0) = \frac{E'_0(0)}{\gamma\sigma_m^3\sqrt{2}}, \quad f'_{3,m}(0) = \frac{1}{\sigma_m^3\gamma^{\frac{1}{2}}2^{\frac{5}{4}}}, \quad f_{2,m}^{(\mathrm{iv})}(0) = \frac{E_0^{(\mathrm{iv})}(0)}{\gamma\sigma_m^3\sqrt{2}}. \end{aligned} \tag{61}$$

The integrals of (59) which satisfy these boundary conditions and which tend to zero as $\xi \to \infty$ are

$$\begin{aligned} f_{2,m}(\xi) = \frac{-E'_0(0)}{2^{\frac{3}{2}}\sigma_m^4\gamma}\left\{e^{-\sigma_m\xi} - 2e^{-\frac{1}{2}\sigma_m\xi}\cos\left(\frac{\sigma_m\xi\sqrt{3}}{2} - \frac{\pi}{3}\right)\right\} \\ + \frac{E_0^{(\mathrm{iv})}(0)}{2^{\frac{3}{2}}\sigma_m^7\gamma}\left\{e^{-\sigma_m\xi} - \frac{2e^{-\frac{1}{2}\sigma_m\xi}}{\sqrt{3}}\cos\left(\frac{\sigma_m\xi\sqrt{3}}{2} + \frac{\pi}{6}\right)\right\} + \mathscr{L}\left(\sigma_m, \frac{E_0^{(\mathrm{vi})}(\xi)}{\gamma\sigma_m^3\sqrt{2}}\right), \\ f_{3,m}(\xi) = \frac{-1}{2^{\frac{9}{4}}\sigma_m^4\gamma^{\frac{1}{2}}}\left\{e^{-\sigma_m\xi} - 2e^{-\frac{1}{2}\sigma_m\xi}\cos\left(\frac{\sigma_m\xi\sqrt{3}}{2} - \frac{\pi}{3}\right)\right\}. \end{aligned}$$

Matching the inner and outer side layers determines the constants C_2, C_3 and b

$$C_2 = \frac{1}{2\gamma\sqrt{2}}\int_0^\infty d\mu \int_\infty^\mu [E_0(\lambda)+\gamma]\,d\lambda, \quad C_3 = 0,$$

$$b = -\frac{1}{2\gamma\sqrt{2}}\int_0^\infty [E_0(\lambda)+\gamma]\,d\lambda - \frac{\epsilon^{\frac{1}{12}}C_2}{\gamma^{\frac{1}{2}}2^{\frac{3}{4}}}.$$

This second stage of the approximation is completed by the evaluation of $D_2(\xi)$, $D_3(\xi)$, $E_2(\xi)$ and $E_3(\xi)$ from the limits of v_2 and v_3 as $z \to \pm\gamma$, and further stages can clearly be obtained by repeating the methods used above.

5. An analysis of convective effects

So far, the transfer of heat has been supposed to be purely conductive. The neglect of convection simplifies the analysis, as the temperature field is found first before the dynamics are discussed.

In this section, we shall study the modifications introduced when some account is taken of the convective transfer of heat. As a first step, we consider the effect of convection as a small perturbation on the previous solutions. To do this, we expand in powers of the small parameters $(\sigma\beta)$

$$T = T_{(0)} + \sigma\beta T_{(1)} + (\sigma\beta)^2 T_{(2)} + \ldots, \tag{62}$$

and similarly for ψ and v. We enclose the subscripts denoting the various terms of the $(\sigma\beta)$-expansion in brackets to avoid confusion with previous subscripts. The subscript zero terms here represent the solutions of either of the last two sections. In this expansion we shall continue to neglect terms in β alone, so that the convective acceleration terms are ignored. The conditions under which this neglect is valid can be established once the analysis of the $(\sigma\beta)$-expansion has been carried out.

The term $T_{(1)}$ in expansion (62) is found by solving the Poisson equation

$$\nabla^2 T_{(1)} = \frac{1}{\epsilon}\frac{\partial\psi_{(0)}}{\partial z}. \tag{63}$$

As Robinson realized, a particular integral of this equation is $T_{(1)} = v_{(0)}$, as is seen by comparing (8) and (63). This particular integral is not the full solution for $T_{(1)}$ as it does not satisfy the boundary conditions on the $T_{(n)}$ which are

$$T_{(n)} = 0 \quad \text{on} \quad x = 0, 1; \quad \frac{\partial T_{(n)}}{\partial z} = 0 \quad \text{on} \quad z = \pm\gamma \quad \text{for} \quad n \geqslant 1. \tag{64}$$

For $T_{(1)}$, we must add to $v_{(0)}$ a harmonic function that vanishes on the sides but whose z derivatives equal $-\partial v_{(0)}/\partial z$ on the top and bottom.

For definiteness, we now restrict our attention to the problem in which all the boundaries are rigid. Then $-\partial v_{(0)}/\partial z = \gamma(2\epsilon)^{-\frac{1}{2}} - 1$ on the top and bottom away from the sides, while inside the side layers it is some more complicated expression also $O(\epsilon^{-\frac{1}{2}})$. The most significant contribution to $T_{(1)}$ is therefore a harmonic function $O(\epsilon^{-\frac{1}{2}})$ vanishing on the sides and with z derivatives of $\gamma(2\epsilon)^{-\frac{1}{2}}$ on top and bottom, and has the representation

$$T_{(1)} = \frac{4\gamma}{\pi^2(2\epsilon)^{\frac{1}{2}}}\sum_{n=0}^{\infty}\frac{\sin(2n+1)\pi x \sinh(2n+1)\pi z}{(2n+1)^2\cosh(2n+1)\pi\gamma}. \tag{65}$$

A different but equivalent expression is quoted by Robinson who also plots isotherms incorporating this small correction term. These are tilted from the vertical, being nearer the cold side at the top and the hot side at the bottom.

It is important for the subsequent developments to consider the way in which the solution for $T_{(1)}$ arises from the right-hand side source term of the Poisson equation (63), and to discuss the structure of $T_{(1)}$ in more detail. The source term vanishes in the interior, is $O(\epsilon^{-1})$ in the Ekman layers and the Ekman extensions of the side layer, and is $O(\epsilon^{-\frac{2}{3}})$ in the side layer itself, these non-vanishing terms being of the characteristic boundary-layer type. It follows that $T_{(1)}$ has components of boundary-layer type, together with an overall component whose scale of variation is that of the whole annulus. The magnitudes of the various boundary-layer components are calculated readily from (63) by requiring a balance of the two sides, and it follows that they are all $O(1)$, just as the particular integral $v_{(0)}$ is in the various regions. The magnitude of the overall component of $T_{(1)}$ is evaluated by calculating the source strengths of each region. To do this for any region, we multiply the magnitude of the right-hand side of (63) by the area of the region. In addition, it must be noted that the effects of sources near the sides are weakened because of the nature of the side boundary condition $T_{(1)} = 0$. Any source within a small distance δ of the side induces an equal and opposite image source at its reflection in the side, and the net effect of this is to reduce the magnitude of the source by a factor δ. The insulating boundary conditions on the top and bottom have the opposite effect of inducing equal image sources, and so there is no corresponding reduction in strength for sources near these boundaries. The source strengths of the regions can therefore be calculated as $O(\epsilon^{-1} \times \epsilon^{\frac{1}{2}}) = O(\epsilon^{-\frac{1}{2}})$ for the Ekman layers, $O(\epsilon^{-\frac{2}{3}} \times \epsilon^{\frac{1}{3}} \times \epsilon^{\frac{1}{3}}) = O(1)$ for the side layers and $O(\epsilon^{-1} \times \epsilon^{\frac{5}{6}} \times \epsilon^{\frac{1}{3}}) = O(\epsilon^{\frac{1}{6}})$ for the Ekman extensions of the side layer. The first of these is predominant and produces the component (65) of $T_{(1)}$.

There is one further addition to be made to the above scheme which we have deduced for the structure of the $T_{(1)}$ field, and this concerns the field due to the sources in the Ekman extensions of the side layers. The overall component of $T_{(1)}$ due to these is $O(\epsilon^{\frac{1}{6}})$ as calculated above, but is of a larger order of magnitude than this in square corner regions of the side layers within distances $O(\epsilon^{\frac{1}{3}})$ of the top and bottom. These regions are intermediate between the source region and its far field. In the interior proper, the source region is point-like, but within the $O(\epsilon^{\frac{1}{3}})$ square corner, it appears as a long line source of strength $O(\epsilon^{-1})$ times the Ekman layer thickness $O(\epsilon^{\frac{1}{2}})$, that is $O(\epsilon^{-\frac{1}{2}})$ per unit length. The effect of this in a region that is $O(\epsilon^{\frac{1}{3}})$ square is $O(\epsilon^{-\frac{1}{6}})$. Although this component of the $T_{(1)}$ field is larger than all the boundary-layer components, it is less than the overall $T_{(1)}$ field and does not turn out to be dynamically significant in the present problem to the lowest order. The successive terms $T_{(n)}$ of our series all have similar components, though $v_{(n)}$ and $\psi_{(n)}$ do not, showing a basic difference between the equations for the $T_{(n)}$'s and those for the dynamical variables.

The next step in our iterative scheme is to solve for $\psi_{(1)}$ and $v_{(1)}$ the first of the sequence of equations

$$\left.\begin{aligned} \epsilon\nabla^2 v_{(n)} - \partial\psi_{(n)}/\partial z &= 0, \\ \epsilon\nabla^4\psi_{(n)} + \partial v_{(n)}/\partial z &= \partial T_{(n)}/\partial x, \end{aligned}\right\} \quad (n \geqslant 1) \qquad \begin{aligned} (66) \\ (67) \end{aligned}$$

subject to the boundary conditions (10). As before, we solve first for the interior and Ekman layers, and later add the necessary side layers. Approximate solutions for general n valid away from the sides are

$$\left.\begin{aligned} v_{(n)} &= \frac{1}{2}\left\{\int_{-\gamma}^{z} + \int_{\gamma}^{z}\right\}\frac{\partial T_{(n)}}{\partial x}dz + \tfrac{1}{2}\{e^{-\zeta_2}\cos\zeta_2 - e^{\zeta_1}\cos\zeta_1\}\int_{-\gamma}^{\gamma}\frac{\partial T_{(n)}}{\partial x}dz, \\ \psi_{(n)} &= \frac{\epsilon^{\frac{1}{2}}}{2\sqrt{2}}\{1 + (\sin\zeta_1 - \cos\zeta_1)e^{\zeta_1} - (\sin\zeta_2 + \cos\zeta_2)e^{-\zeta_2}\}\int_{-\gamma}^{\gamma}\frac{\partial T_{(n)}}{\partial x}dz, \end{aligned}\right\} \tag{68}$$

and these expressions are known once $T_{(n)}$ is known for the interior. The errors here are $O(\epsilon^{\frac{1}{2}})$ compared with terms retained, and the solutions (68) reduce to the subscript zero solution (12) and (13) when we set $T_{(n)} = T_{(0)} = x$.

When we come to consider the solutions for the side boundary layers, one significant point of difference between the solutions for $\psi_{(1)}$ and $v_{(1)}$ compared with $\psi_{(0)}$ and $v_{(0)}$ for this problem of all rigid boundaries is that the subscript one terms must have side boundary layers of double structure. This double structure is avoided with the subscript zero terms because the interior solution for $\psi_{(0)}$ is symmetric and $v_{(0)}$ antisymmetric in z. Although $\psi_{(n)}$ in (68) is symmetric in z, $v_{(n)}$ is not antisymmetric in z unless $\partial T_{(n)}/\partial x$ is symmetric in z, and $\partial T_{(1)}/\partial x$ is not symmetric in z as is readily seen from its leading term (65).

The most significant term in $T_{(1)}$ is the $O(\epsilon^{-\frac{1}{2}})$ overall field and this drives an interior flow for which $v_{(1)}$ is $O(\epsilon^{-\frac{1}{2}})$ and $\psi_{(1)}$ is $O(1)$. It is also necessary that $v_{(1)}$ and $\psi_{(1)}$ are of the same magnitudes in the outer side layer, its Ekman extensions and the extensions of the inner side layer, but $\psi_{(1)}$ is $O(\epsilon^{-\frac{1}{6}})$ while $v_{(1)}$ is still $O(\epsilon^{-\frac{1}{2}})$ in the inner side layer. In the side layers proper, the temperature gradient term $\partial T_{(1)}/\partial x$ arising from the overall field is $O(\epsilon^{-\frac{1}{2}})$ and is of equal importance with the most significant velocity terms, but the temperature gradient arising from the term in $T_{(1)}$ of side boundary-layer character is less significant as it is $O(\epsilon^{-\frac{1}{3}})$. The temperature gradient term is not significant in the Ekman extensions, and the $O(\epsilon^{-\frac{1}{6}})$ field in the $O(\epsilon^{\frac{1}{3}})$ square corner region does not induce any dynamical motion as far as the largest components are concerned.

Having laid the groundwork, it is now possible to deduce the forms of the general terms $T_{(n)}$, $v_{(n)}$ and $\psi_{(n)}$, and thus the structure of the expansion in powers of $\sigma\beta$. The general term $T_{(n)}$ satisfies the equation

$$\nabla^2 T_{(n)} = \frac{1}{\epsilon}\sum_{i=0}^{n-1}\left\{\frac{\partial\psi_{(i)}}{\partial z}\frac{\partial T_{(n-1-i)}}{\partial x} - \frac{\partial\psi_{(i)}}{\partial x}\frac{\partial T_{(n-1-i)}}{\partial z}\right\}. \tag{69}$$

It can be verified by induction that, for general n, $T_{(n)}$ has a term $O(\epsilon^{-\frac{1}{2}n})$ with overall length scale, contributed to this order by the right-hand side sources in the interior and main Ekman layers but not the sides. It has less significant boundary-layer terms which are $O(\epsilon^{-\frac{1}{2}n+\frac{1}{2}})$ for the inner and outer side layers and their Ekman extensions. There are also $O(\epsilon^{-\frac{1}{2}n+\frac{1}{3}})$ terms in the $O(\epsilon^{\frac{1}{3}})$ square corner regions of the type found above for $T_{(1)}$, and similar terms for $O(\epsilon^{\frac{1}{4}})$ square corner regions now that there is also an outer side layer that are $O(\epsilon^{-\frac{1}{2}n+\frac{1}{4}})$. To conform with this pattern, $v_{(n)}$ is everywhere $O(\epsilon^{-\frac{1}{2}n})$, and $\psi_{(n)}$ is $O(\epsilon^{-\frac{1}{2}n+\frac{1}{2}})$ everywhere except in the inner side layer proper where it is $O(\epsilon^{-\frac{1}{2}n+\frac{1}{3}})$. That this scheme is self-consistent can be verified in a straightforward manner, though it should be noted that in calculating $\partial T_{(n)}/\partial z$ for $n \geqslant 1$ for the side layers, the most significant term is that arising from the overall field, but this is not $O(\epsilon^{-\frac{1}{2}n})$ since $T_{(n)} = 0$ at both sides. This gradient is therefore $O(\epsilon^{-\frac{1}{2}n+\frac{1}{3}})$ in the inner side layer and $O(\epsilon^{-\frac{1}{2}n+\frac{1}{4}})$ in the outer side layer.

The scheme we have just deduced shows clearly that our expansion is essentially one in powers of the parameter $\sigma\beta\epsilon^{-\frac{1}{2}}$, and it demonstrates the form of the flow when $\sigma\beta\epsilon^{-\frac{1}{2}}$ becomes of order unity, but β is still small. To study this, let us set $\lambda = \sigma\beta\epsilon^{-\frac{1}{2}}$ so that the governing equations with convective acceleration terms still neglected are

$$\epsilon\nabla^2 v = \partial\psi/\partial z, \tag{70}$$

$$\epsilon\nabla^4\psi + \partial v/\partial z = \partial T/\partial x, \tag{71}$$

$$\epsilon^{\frac{1}{2}}\nabla^2 T = \lambda\left(\frac{\partial\psi}{\partial z}\frac{\partial T}{\partial x} - \frac{\partial\psi}{\partial x}\frac{\partial T}{\partial z}\right). \tag{72}$$

When no restriction is placed on the size of λ, our series analysis suggests that we look for a solution for which v is everywhere $O(1)$, and ψ is $O(\epsilon^{\frac{1}{2}})$ everywhere except in the inner side layers where it is $O(\epsilon^{\frac{1}{3}})$. The temperature T should have an $O(1)$ term of overall scale, $O(\epsilon^{\frac{1}{2}})$ terms for all boundary layers, and $O(\epsilon^{\frac{1}{3}})$ and $O(\epsilon^{\frac{1}{4}})$ terms for the two kinds of square corner regions. A solution of this kind can be found as will be shown.

We shall first discuss the solution in the interior, for which it follows from (70) that

$$\psi = \epsilon^{\frac{1}{2}} f(x) \tag{73}$$

for some function f. (Here and subsequently, we neglect terms which are $O(\epsilon^{\frac{1}{2}})$ compared with those retained.) There is therefore a small upward flow in the interior but no radial flow. Using T_i to denote the largest overall component of the temperature field, then

$$\nabla^2 T_i = -\lambda f'(x)(\partial T_i/\partial z), \tag{74}$$

to the lowest order in ϵ.

Expressions relating the variables v and ψ to T_i valid uniformly across the interior and Ekman layers away from the sides are given by (68) with the subscript (n) replaced by i. For consistency with (73) therefore, the overall temperature field must satisfy the constraint

$$f(x) = \frac{1}{2\sqrt{2}}\int_{-\gamma}^{\gamma}\frac{\partial T_i}{\partial x}\,dz. \tag{75}$$

The flow in the interior region is solved once T_i and $f(x)$ are known. To complete the specification of the problem for T_i, we need boundary conditions. Since the interior temperature field is larger in magnitude than any of the terms of boundary layer type, it must itself satisfy the boundary conditions on T at the sides, so that

$$T_i = 0 \quad \text{at} \quad x = 0, \quad T_i = 1 \quad \text{at} \quad x = 1. \tag{76}$$

It follows that $f(x)$ must satisfy the constraint

$$\sqrt{2}\int_0^1 f(x)\,dx = \gamma, \tag{77}$$

which is obtained by integrating relation (75) with respect to x across the annulus.

The boundary conditions on T_i at $z = \pm\gamma$ require more discussion since, in evaluating $\partial T/\partial z$ here, the contribution from the $O(\epsilon^{\frac{1}{2}})$ Ekman layer term, which we shall write as $\epsilon^{\frac{1}{2}}T_e(x, \zeta)$, is equally as important as $\partial T_i/\partial z$. Using this notation, the leading terms in (72) applied to the Ekman layers are

$$\frac{\partial^2 T_e}{\partial \zeta^2} = \lambda \left(\frac{2}{\epsilon}\right)^{\frac{1}{2}} \frac{\partial \psi}{\partial \zeta}\frac{\partial T_i}{\partial x}. \tag{78}$$

This equation can be integrated once to give

$$\frac{\partial T_e}{\partial \zeta} = \lambda \left(\frac{2}{\epsilon}\right)^{\frac{1}{2}} \frac{\partial T_i}{\partial x}(\psi - \epsilon^{\frac{1}{2}}f(x)), \tag{79}$$

where the condition that $T_e \to 0$ and $\psi \to \epsilon^{\frac{1}{2}}f(x)$ away from the Ekman layers has been applied. The temperature boundary conditions at the top and bottom therefore become

$$\frac{\partial T}{\partial z} = \frac{\partial T_i}{\partial z} + \frac{1}{\sqrt{2}}\frac{\partial T_e}{\partial \zeta} = \frac{\partial T_i}{\partial z} - \lambda f(x)\frac{\partial T_i}{\partial x} = 0 \quad \text{at} \quad z = \pm\gamma, \tag{80}$$

so that the boundary conditions on T and T_i differ. This reflects the fact that there is significant convection in the Ekman layers, and that the isotherms, which are vertical at the top and bottom boundaries, are turned through finite angles in the Ekman layers.

The problem of determining the interior flow has now been formulated as that of solving equation (74) for T_i subject to boundary conditions (76) and (80) with f unknown, and then using the constraint (75) to evaluate f. This is a non-linear problem and it is unlikely that a general solution is readily obtainable. Most probably a numerical treatment will be necessary to obtain solutions for general values of λ though, as we shall show below, a series development in powers of λ can be carried sufficiently far to give interesting results concerning modifications introduced by convection.

Before we start on this development a general property of the flow will be derived. (74) can be integrated with respect to z between $z = \pm\gamma$ using the boundary conditions (80) and relation (75) to give

$$2\sqrt{2}f'(x) + \lambda f(x)\frac{d(\delta T)}{dx} = -\lambda f'(x)\,\delta T, \tag{81}$$

where we introduce the notation $\delta T = T_i(z = \gamma) - T_i(z = -\gamma)$ for the difference in temperature between the top and the bottom of the annulus at the horizontal position x. Relation (81) can be integrated to give

$$(2\sqrt{2} + \lambda\,\delta T)f(x) = \text{constant} = 2\sqrt{2}f(0) = 2\sqrt{2}f(1), \tag{82}$$

the latter two equalities being consequences of the fact that the side boundary conditions of constant temperature make δT vanish there. Relation (82) can readily be interpreted in terms of the transfer of heat in the horizontal direction between the two vertical walls. There are both conductive and convective components of the horizontal transfer, though convective transfer takes place only in the Ekman layers since there is no radial flow velocity in the interior. There is of course vertical convective transfer of heat in the interior. The heat transferred by conduction, per unit time and unit length in the y-direction, in the negative x-direction at any value of x is

$$\rho_0 C \int_{-\frac{1}{2}d}^{\frac{1}{2}d} \kappa \frac{\partial T'}{\partial x'} dz' = \rho_0 C\kappa \Delta T \int_{-\gamma}^{\gamma} \frac{\partial T}{\partial x} dz = 2\sqrt{2}\rho_0 C\kappa \Delta T f(x), \tag{83}$$

where C is the specific heat. The Ekman layer flux is of non-dimensional amount $\epsilon^{\frac{1}{2}} f(x)$ in the negative x-direction in the upper layer and the opposite in the lower layer. The horizontal flux in the Ekman layers varies with position since there is inflow and outflow in the vertical direction. There is a net convective transfer of heat in the negative x-direction due to the difference in temperature between fluid in the two Ekman layers and this is of dimensional amount

$$\epsilon^{\frac{1}{2}} f(x)\,\delta T \frac{\rho_0 C \alpha g (\Delta T)^2 \ell}{2\Omega} = \rho_0 C\kappa \Delta T \,\lambda f(x)\,\delta T. \tag{84}$$

(81) therefore expresses the fact that the total heat transfer obtained by summing the two contributions is the same at all values of x, and is the same as that transferred by conduction across the sides. An expression for the Nusselt number, the ratio of the actual heat transfer to that which would obtain with conductive transfer only when $T = x$ and $f(x) = \gamma/\sqrt{2}$, is

$$N = 2^{\frac{1}{2}} f(0)/\gamma. \tag{85}$$

We shall next evaluate the early stages of the expansions for T_i and $f(x)$ in powers of λ, which can be done in an iterative manner. This is equivalent to evaluating the most significant contributions to the terms of the series (62) at the start of this section. It is readily seen that

$$T_i = x + \frac{2\lambda\gamma 2^{\frac{1}{2}}}{\pi^2} \sum_{n=0}^{\infty} \frac{\sin(2n+1)\pi x \sinh(2n+1)\pi z}{(2n+1)^2 \cosh(2n+1)\pi\gamma} + O(\lambda^2), \tag{86}$$

$$f(x) = \frac{\gamma}{2^{\frac{1}{2}}} + \frac{\gamma^2\lambda^2 2^{\frac{1}{2}}}{\pi^2} \left\{ \frac{2}{\pi} \sum_{n=0}^{\infty} \frac{\tanh(2n+1)\pi\gamma}{(2n+1)^3} - \sum_{n=0}^{\infty} \frac{\sin(2n+1)\pi x \tanh(2n+1)\pi\gamma}{(2n+1)^2} \right\} + O(\lambda^4).$$

The $O(\lambda)$ term in T_i is just the correction term (65) and the $O(\lambda^2)$ contribution to f can be calculated directly from this via relation (82) without going to the next order in the expansion for T_i. It is also necessary to use condition (77) to find the $O(\lambda^2)$ contribution of $f(0)$ and complete the calculation. Successive terms in the T_i expansions involve successively more series summations and the $O(\lambda^2)$ and $O(\lambda^3)$ terms, which are also harmonic functions and can readily be derived, are not quoted. They are respectively symmetric and antisymmetric functions of z, which explains why there is no $O(\lambda^3)$ term in $f(x)$. The $O(\lambda^4)$ term of T_i is of the first in the development for which the right-hand side of (74) is significant. The $O(\lambda^4)$ contribution to $f(0)$ can be found using the $O(\lambda^3)$ term of T_i and the relations (82) and (77), and gives a further term in the expansion for the Nusselt number. It is found that

$$N = 1 + \lambda^2 G(\gamma) + \lambda^4 H(\gamma) + O(\lambda^5), \tag{87}$$

where
$$G(\gamma) = \frac{4\gamma}{\pi^3} \sum_{n=0}^{\infty} \frac{\tanh \pi\gamma(2n+1)}{(2n+1)^3},$$
$$H(\gamma) = G^2(\gamma) - \frac{2\gamma^2}{\pi^4} \sum_{n=0}^{\infty} \frac{\tanh^2 (2n+1)\pi\gamma}{(2n+1)^4} - \frac{32\gamma^2}{\pi^6} \sum_{n=1}^{\infty} \{1 + 2n\pi\gamma \coth 2n\pi\gamma\} S_n^2(\gamma),$$
$$S_n(\gamma) = \sum_{m=0}^{\infty} \frac{\tanh (2m+1)\pi\gamma}{(2m+1)[(2m+1)^2 - 4n^2]}.$$

Graphs of the functions G and $-10H$ which depend on the geometrical parameter γ are shown in figure 2. The function G is positive except in the limiting case $\gamma = 0$ of a very shallow annulus, and tends rapidly to its asymptotic form $0{\cdot}136\gamma$ valid when γ is large, being within 10 % of this value even when the annulus is square. Thus, as is to be expected, the transfer of heat is aided by convection.

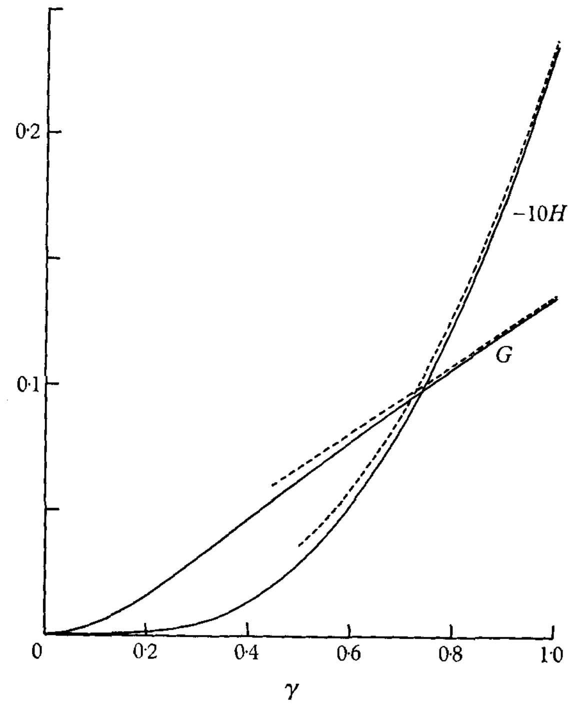

FIGURE 2. The functions $G(\gamma)$ and $-10\,H(\gamma)$ for expansion (87) of the Nusselt number. The dashed curves represent the asymptotic expressions valid for large γ quoted in the text.

The growth of the Nusselt number with λ is modified by the λ^4 term which is always negative except when $\gamma = 0$. The asymptotic form of H for large γ is $-0{\cdot}00484\gamma^2 - 0{\cdot}01897\gamma^3$, and is also closely attained for moderate values of γ.

Differentiating (86) shows that the vertical velocity
$$w = -\frac{\partial\psi}{\partial x} = -\epsilon^{\frac{1}{2}}\gamma^2\lambda^2 \left\{ -\frac{\sqrt{2}}{\pi} \sum_{n=0}^{\infty} \frac{\cos (2n+1)\pi x \tanh (2n+1)\pi\gamma}{(2n+1)} \right\} + O(\lambda^4\epsilon^{\frac{1}{2}}). \quad (88)$$

The profiles of minus the vertical velocity in the interior as given by the series in curly brackets in (88) are displayed in figure 3 for different values of γ. The series is antisymmetric about $x = \frac{1}{2}$ and so its sum in the range $0{\cdot}5 \leqslant x \leqslant 1$ only is plotted. In all cases, there is a downflow in the half of the annulus near the hot wall, and an upflow in the other half. These flows are additional to the flows in the side boundary layers found in §3. The function represented by the series in (88) has logarithmic singularities near $x = 0$ and $x = 1$, and has the explicit sum

$$-(1/\pi\sqrt{2})\log\cot(\tfrac{1}{2}\pi x)$$

in the limit $\gamma \to \infty$ (Jeffreys & Jeffreys 1956, p. 439).

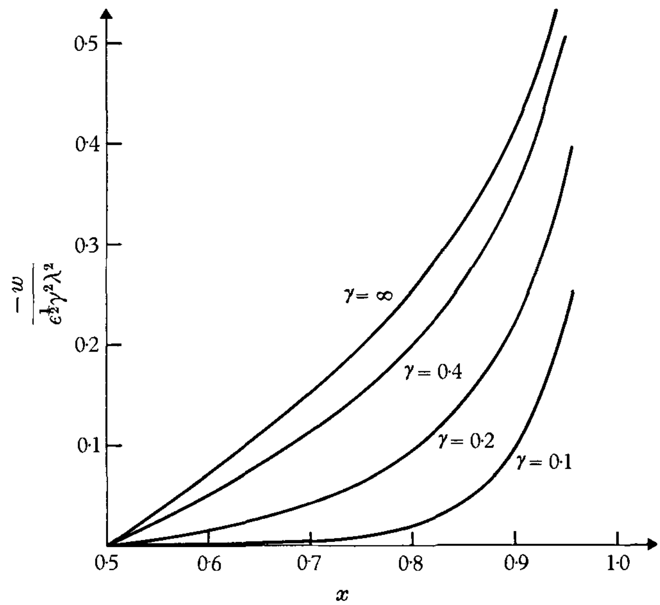

FIGURE 3. Profiles of minus the vertical velocity in the interior for different values of γ.

Having discussed the solution of equations (70) to (72) for the interior, and supposing T_i and f to have been determined, the side boundary layers necessary to complete the flow can be calculated following previous analysis. The leading terms of the solutions for the dynamical variables v and ψ in the outer side layer near $x = 0$ are

$$\left.\begin{aligned} v &= -\sqrt{2}f(0) + \int_{-\gamma}^{z} \left(\frac{\partial T_i}{\partial x}\right)_{x=0} dz - k\exp\left[\frac{-\eta}{\gamma^{\frac{1}{2}}2^{\frac{1}{4}}}\right], \\ \psi &= \epsilon^{\frac{1}{2}}\left\{f(0) + \frac{kz}{\gamma\sqrt{2}}\exp\left[\frac{-\eta}{\gamma^{\frac{1}{2}}2^{\frac{1}{4}}}\right]\right\}, \end{aligned}\right\} \tag{89}$$

where
$$k = f(0)\sqrt{2} + \frac{1}{2\gamma}\int_{-\gamma}^{\gamma}(z-\gamma)\left(\frac{\partial T_i}{\partial x}\right)_{x=0} dz.$$

For the inner side layer near $x = 0$

$$\left.\begin{aligned} v &= -k - \frac{k\epsilon^{\frac{1}{12}}\xi}{\gamma^{\frac{1}{2}}2^{\frac{1}{4}}} + \int_{-\gamma}^{z}\left(\frac{\partial T_i}{\partial x}\right)_{x=0} dz \\ &\quad + \frac{1}{2}\sum_{m=1}^{\infty}\alpha_m\left\{e^{-\sigma_m\xi} + \frac{2}{\sqrt{3}}e^{-\frac{1}{2}\sigma_m\xi}\cos\left(\frac{\sigma_m\xi\sqrt{3}}{2} - \frac{\pi}{6}\right)\right\}\cos\sigma_m^3(z+\gamma), \\ \psi &= \epsilon^{\frac{1}{3}}\sum_{m=1}^{\infty}\frac{\alpha_m}{2\sigma_m}\left\{e^{-\sigma_m\xi} - \frac{2}{\sqrt{3}}e^{-\frac{1}{2}\sigma_m\xi}\cos\left(\frac{\sigma_m\xi\sqrt{3}}{2} + \frac{\pi}{6}\right)\right\}\sin\sigma_m^3(z+\gamma), \end{aligned}\right\} \tag{90}$$

where $$\alpha_m = -\frac{1}{\gamma}\int_{-\gamma}^{\gamma} dz \cos\sigma_m^3(z+\gamma)\left[\int_{-\gamma}^{z}\left(\frac{\partial T_i}{\partial x}\right)_{x=0} dz\right].$$

Similar expressions for the layers near $x = 1$ may easily be found and the solutions for the Ekman extensions of the side layers are readily evaluated, their ζ variation being of the characteristic kind as in expressions (68), for instance. It is also easy to find partial expressions for the components of the temperature field of boundary-layer character in the side layers and their Ekman extensions, but for the full matching and determination of these, it is necessary to solve full partial differential equations in the square-corner regions described above. We shall not determine these small corrections to the temperature field.

Having set up this problem in which conductive and convective terms are of comparable magnitude and shown the nature of the solution, we can re-examine the approximation we made earlier of neglecting the convective acceleration terms in the equations of motion. By comparing the magnitudes of terms neglected with those retained, it is seen that this approximation is least good in the inner side layers and their Ekman extensions where the error is $O(\beta\epsilon^{-\frac{1}{3}})$. Th theory we have outlined therefore requires

$$\epsilon \ll 1, \quad \beta\epsilon^{-\frac{1}{3}} \ll 1, \quad \sigma\beta\epsilon^{-\frac{1}{2}} \leqslant O(1). \tag{91}$$

There is unfortunately a lack of experimental data for annular flows with the upper surface rigid against which to test any of this theory as most of the experiments are done with the upper surface free. As the subsequent discussion shows, the theoretical extensions to include convective effects in the free surface problem are not readily accomplished in the same way as we have treated the rigid surface problem. If, for want of anything more appropriate, one refers to the experimental work of Fowlis & Hide for the free surface problem, it is seen that the parameter range defined by expressions (91) covers a good section of the lower symmetrical régime and includes part of the line of transition from axisymmetrical flow to the steady wave régime.

Finally, we consider briefly what happens when we tackle the problem with the upper surface free in the same manner. Unfortunately, it appears that it is not possible to achieve results comparable with those we were able to obtain for the rigid surface case.

In the free surface problem the effect of the more complicated side boundary layers becomes dominant when we develop the series expansion in powers of $(\sigma\beta)$ as before. This becomes apparent right away in the solution for $T_{(1)}$. The source strengths of the Ekman layers, the inner and outer side layers and the Ekman extensions of the outer side layers are all of equal significance in producing an $O(1)$ overall field for $T_{(1)}$. (Note that the particular integral $v_{(0)}$ of $T_{(1)}$ now satisfies the correct boundary condition on the upper surface. It also satisfies the correct boundary condition on the lower surface except near the sides. The error within $O(\epsilon^{\frac{1}{4}})$ of the sides is $O(\epsilon^{-\frac{1}{2}})$, however, and this induces an $O(1)$ component of $T_{(1)}$ in the overall field which is of the same magnitude as $v_{(0)}$.) In all the side layers and the Ekman extensions, there are also $O(1)$ components of $T_{(1)}$ with boundary-layer character while, in the square $O(\epsilon^{\frac{1}{3}})$ and $O(\epsilon^{\frac{1}{4}})$ regions near the bottom, there are components of $T_{(1)}$ which are locally $O(\epsilon^{-\frac{1}{6}})$ and $O(\epsilon^{-\frac{1}{4}})$ respectively. (Note that these terms arise only in the lower corners. The meridional flow in the upper corners is weaker because of the free surface boundary condition, as is shown by the solutions (43) and (47).) The first correction to the temperature field is therefore much larger near these lower corners than it is elsewhere. This situation is accentuated when we go on to discuss $T_{(2)}$, for the dominant source region is then the lower square $O(\epsilon^{\frac{1}{4}})$ corners which produce a dominant $O(\epsilon^{-\frac{1}{2}})$ component of $T_{(2)}$ and have a local term which is $O(\epsilon^{-\frac{3}{4}})$. The expansion developed thus far gives an interior solution for T which is

$$O(1)+O(\sigma\beta)+O\left(\frac{\sigma^2\beta^2}{\epsilon^{\frac{1}{2}}}\right)+\dots.$$

In the $O(\epsilon^{\frac{1}{4}})$ corner, the local temperature field is

$$O(1)+O\left(\frac{\sigma\beta}{\epsilon^{\frac{1}{4}}}\right)+O\left(\frac{\sigma^2\beta^2}{\epsilon^{\frac{3}{4}}}\right)+\dots,$$

while in the $O(\epsilon^{\frac{1}{3}})$ corner, the local temperature field is

$$O(1)+O\left(\frac{\sigma\beta}{\epsilon^{\frac{1}{6}}}\right)+O\left(\frac{\sigma^2\beta^2}{\epsilon^{\frac{2}{3}}}\right)+\dots.$$

It is apparent therefore that we do not have a solution of such simple structure as we had previously when all quantities had series expansions in powers of $\sigma\beta\epsilon^{-\frac{1}{2}}$. The ratio of the third to the second terms of the three series just quoted is $\sigma\beta\epsilon^{-\frac{1}{2}}$ and, if we set this to be of order unity, then our series apparently all converge. The justification for this last remark is that it is possible to construct a self-consistent scheme for the magnitudes of the various variables in the different regions. This has

$$\begin{aligned} v &= O(1) \text{ throughout,}\\ \psi &= O(\epsilon) \text{ in the interior and its Ekman layers,}\\ \psi &= O(\epsilon^{\frac{1}{3}}) \text{ in the inner side layers,}\\ \psi &= O(\epsilon^{\frac{1}{2}}) \text{ elsewhere near the sides,}\\ T &= x+O(\epsilon^{\frac{1}{2}})+O(\epsilon^{\frac{1}{4}}) \text{ in the lower } \epsilon^{\frac{1}{4}} \text{ square corners,}\\ &\qquad\qquad +O(\epsilon^{\frac{1}{3}}) \text{ in the lower } \epsilon^{\frac{1}{3}} \text{ square corners.} \end{aligned}$$

Then the dynamical problem of determining v and ψ uncouples from that of determining the corrections to $T = x$, and the solution is identical with that of § 4. The only novel feature of this problem marking a step forward from the problem of § 4 is that, when we come to write down equation (72) for the determining of the correction to the conductive temperature field in the lower square corner regions, terms on both sides are of equal significance. Since, in this scheme, the corrections to the conductive temperature field are everywhere small, there does not appear to be much to be gained from continuing this line of investigation. Presumably the solution to the free surface problem when conductive and convective effects are of equal significance is so different from that when conductive effects are dominant that it can not be approached in this way, and in this respect it is different from the problem in which all the boundaries are rigid.

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